

# MAA AMC

American Mathematics Competitions

American Mathematics Competitions

69th Annual

# AMC 12A

American Mathematics Competition 12A

Wednesday, February 7, 2018

## INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR COMPETITION MANAGER TELLS YOU.
2. This is a 25-question multiple-choice exam. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 12 Answer Sheet with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded. **You must use and submit the original answer sheets provided by the MAA AMC. Photocopies will not be scored.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the exam will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the exam, your competition manager will ask you to record certain information on the answer form.
8. When your competition manager gives the signal, begin working on the problems. You will have **75 minutes** to complete the exam.
9. When you finish the exam, *sign your name* in the space provided at the top of the Answer Sheet.

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The Committee on the American Mathematics Competitions reserves the right to disqualify scores from a school if it determines that the required security procedures were not followed.

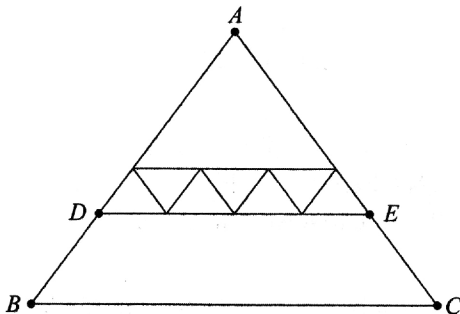
*Students who score well on this AMC 12 will be invited to take the 36th annual American Invitational Mathematics Examination (AIME) on Tuesday, March 6, 2018 or Wednesday, March 21, 2018. More details about the AIME are on the back page of this test booklet.*

1. A large urn contains 100 balls, of which 36% are red and the rest are blue. How many of the blue balls must be removed so that the percentage of red balls in the urn will be 72%? (No red balls are to be removed.)
- (A) 28      (B) 32      (C) 36      (D) 50      (E) 64
2. While exploring a cave, Carl comes across a collection of 5-pound rocks worth \$14 each, 4-pound rocks worth \$11 each, and 1-pound rocks worth \$2 each. There are at least 20 of each size. He can carry at most 18 pounds. What is the maximum value, in dollars, of the rocks he can carry out of the cave?
- (A) 48      (B) 49      (C) 50      (D) 51      (E) 52
3. How many ways can a student schedule 3 mathematics courses—algebra, geometry, and number theory—in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.)
- (A) 3      (B) 6      (C) 12      (D) 18      (E) 24
4. Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, “We are at least 6 miles away,” Bob replied, “We are at most 5 miles away.” Charlie then remarked, “Actually the nearest town is at most 4 miles away.” It turned out that none of the three statements was true. Let  $d$  be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of  $d$ ?
- (A)  $(0, 4)$       (B)  $(4, 5)$       (C)  $(4, 6)$       (D)  $(5, 6)$       (E)  $(5, \infty)$
5. What is the sum of all possible values of  $k$  for which the polynomials  $x^2 - 3x + 2$  and  $x^2 - 5x + k$  have a root in common?
- (A) 3      (B) 4      (C) 5      (D) 6      (E) 10
6. For positive integers  $m$  and  $n$  such that  $m + 10 < n + 1$ , both the mean and the median of the set  $\{m, m + 4, m + 10, n + 1, n + 2, 2n\}$  are equal to  $n$ . What is  $m + n$ ?
- (A) 20      (B) 21      (C) 22      (D) 23      (E) 24

7. For how many (not necessarily positive) integer values of  $n$  is the value of  $4000 \cdot \left(\frac{2}{5}\right)^n$  an integer?

(A) 3      (B) 4      (C) 6      (D) 8      (E) 9

8. All of the triangles in the diagram below are similar to isosceles triangle  $ABC$ , in which  $AB = AC$ . Each of the 7 smallest triangles has area 1, and  $\triangle ABC$  has area 40. What is the area of trapezoid  $DBCE$ ?



(A) 16      (B) 18      (C) 20      (D) 22      (E) 24

9. Which of the following describes the largest subset of values of  $y$  within the closed interval  $[0, \pi]$  for which

$$\sin(x + y) \leq \sin(x) + \sin(y)$$

for every  $x$  between 0 and  $\pi$ , inclusive?

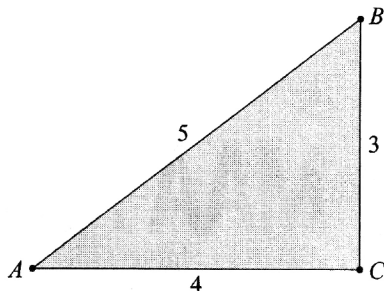
(A)  $y = 0$       (B)  $0 \leq y \leq \frac{\pi}{4}$       (C)  $0 \leq y \leq \frac{\pi}{2}$       (D)  $0 \leq y \leq \frac{3\pi}{4}$   
(E)  $0 \leq y \leq \pi$

10. How many ordered pairs of real numbers  $(x, y)$  satisfy the following system of equations?

$$\begin{aligned}x + 3y &= 3 \\|x| - |y| &= 1\end{aligned}$$

(A) 1      (B) 2      (C) 3      (D) 4      (E) 8

11. A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point  $A$  falls on point  $B$ . What is the length in inches of the crease?



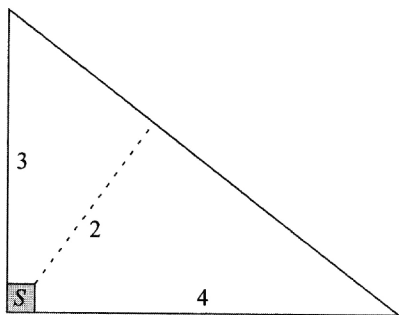
- (A)  $1 + \frac{1}{2}\sqrt{2}$     (B)  $\sqrt{3}$     (C)  $\frac{7}{4}$     (D)  $\frac{15}{8}$     (E) 2
12. Let  $S$  be a set of 6 integers taken from  $\{1, 2, \dots, 12\}$  with the property that if  $a$  and  $b$  are elements of  $S$  with  $a < b$ , then  $b$  is not a multiple of  $a$ . What is the least possible value of an element of  $S$ ?
- (A) 2    (B) 3    (C) 4    (D) 5    (E) 7
13. How many nonnegative integers can be written in the form  $a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0$ , where  $a_i \in \{-1, 0, 1\}$  for  $0 \leq i \leq 7$ ?
- (A) 512    (B) 729    (C) 1094    (D) 3281    (E) 59,048
14. The solution to the equation  $\log_{3x} 4 = \log_{2x} 8$ , where  $x$  is a positive real number other than  $\frac{1}{3}$  or  $\frac{1}{2}$ , can be written as  $\frac{p}{q}$ , where  $p$  and  $q$  are relatively prime positive integers. What is  $p + q$ ?
- (A) 5    (B) 13    (C) 17    (D) 31    (E) 35
15. A scanning code consists of a  $7 \times 7$  grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of  $90^\circ$  counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?
- (A) 510    (B) 1022    (C) 8190    (D) 8192    (E) 65,534



16. Which of the following describes the set of values of  $a$  for which the curves  $x^2 + y^2 = a^2$  and  $y = x^2 - a$  in the real  $xy$ -plane intersect at exactly 3 points?

(A)  $a = \frac{1}{4}$     (B)  $\frac{1}{4} < a < \frac{1}{2}$     (C)  $a > \frac{1}{4}$     (D)  $a = \frac{1}{2}$   
 (E)  $a > \frac{1}{2}$

17. Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths of 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square  $S$  so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from  $S$  to the hypotenuse is 2 units. What fraction of the field is planted?



- (A)  $\frac{25}{27}$     (B)  $\frac{26}{27}$     (C)  $\frac{73}{75}$     (D)  $\frac{145}{147}$     (E)  $\frac{74}{75}$
18. Triangle  $ABC$  with  $AB = 50$  and  $AC = 10$  has area 120. Let  $D$  be the midpoint of  $\overline{AB}$ , and let  $E$  be the midpoint of  $\overline{AC}$ . The angle bisector of  $\angle BAC$  intersects  $\overline{DE}$  and  $\overline{BC}$  at  $F$  and  $G$ , respectively. What is the area of quadrilateral  $FDBG$ ?
- (A) 60    (B) 65    (C) 70    (D) 75    (E) 80
19. Let  $A$  be the set of positive integers that have no prime factors other than 2, 3, or 5. The infinite sum
- $$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{16} + \frac{1}{18} + \frac{1}{20} + \cdots$$
- of the reciprocals of all the elements of  $A$  can be expressed as  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. What is  $m+n$ ?
- (A) 16    (B) 17    (C) 19    (D) 23    (E) 36

20. Triangle  $ABC$  is an isosceles right triangle with  $AB = AC = 3$ . Let  $M$  be the midpoint of hypotenuse  $\overline{BC}$ . Points  $I$  and  $E$  lie on sides  $\overline{AC}$  and  $\overline{AB}$ , respectively, so that  $AI > AE$  and  $AIME$  is a cyclic quadrilateral. Given that triangle  $EMI$  has area 2, the length  $CI$  can be written as  $\frac{a-\sqrt{b}}{c}$ , where  $a$ ,  $b$ , and  $c$  are positive integers and  $b$  is not divisible by the square of any prime. What is the value of  $a + b + c$ ?
- (A) 9      (B) 10      (C) 11      (D) 12      (E) 13
21. Which of the following polynomials has the greatest real root?
- (A)  $x^{19} + 2018x^{11} + 1$       (B)  $x^{17} + 2018x^{11} + 1$   
(C)  $x^{19} + 2018x^{13} + 1$       (D)  $x^{17} + 2018x^{13} + 1$   
(E)  $2019x + 2018$
22. The solutions to the equations  $z^2 = 4 + 4\sqrt{15}i$  and  $z^2 = 2 + 2\sqrt{3}i$ , where  $i = \sqrt{-1}$ , form the vertices of a parallelogram in the complex plane. The area of this parallelogram can be written in the form  $p\sqrt{q} - r\sqrt{s}$ , where  $p$ ,  $q$ ,  $r$ , and  $s$  are positive integers and neither  $q$  nor  $s$  is divisible by the square of any prime number. What is  $p + q + r + s$ ?
- (A) 20      (B) 21      (C) 22      (D) 23      (E) 24
23. In  $\triangle PAT$ ,  $\angle P = 36^\circ$ ,  $\angle A = 56^\circ$ , and  $PA = 10$ . Points  $U$  and  $G$  lie on sides  $\overline{TP}$  and  $\overline{TA}$ , respectively, so that  $PU = AG = 1$ . Let  $M$  and  $N$  be the midpoints of segments  $\overline{PA}$  and  $\overline{UG}$ , respectively. What is the degree measure of the acute angle formed by lines  $MN$  and  $PA$ ?
- (A) 76      (B) 77      (C) 78      (D) 79      (E) 80
24. Alice, Bob, and Carol play a game in which each of them chooses a real number between 0 and 1. The winner of the game is the one whose number is between the numbers chosen by the other two players. Alice announces that she will choose her number uniformly at random from all the numbers between 0 and 1, and Bob announces that he will choose his number uniformly at random from all the numbers between  $\frac{1}{2}$  and  $\frac{2}{3}$ . Armed with this information, what number should Carol choose to maximize her chance of winning?
- (A)  $\frac{1}{2}$       (B)  $\frac{13}{24}$       (C)  $\frac{7}{12}$       (D)  $\frac{5}{8}$       (E)  $\frac{2}{3}$